Quantum Cloning Machines of a d-level System

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The optimal N to M ($M > N$) quantum cloning machines for the d-level system are presented. The unitary cloning transformations achieve the bound of the fidelity.

No-cloning theorem is one of the most fundamental differences between classical and quantum information theories. It tells us that an unknown quantum state can not be copied perfectly [1]. This no-cloning theorem has important consequences for the whole quantum information processing [2]. However, no-cloning theorem does not forbid imperfect cloning, and it is interesting to know how well we can copy an unknown quantum state. Bužek and Hillery [3] introduced a universal quantum cloning machine (UQCM) for an arbitrary pure input state. It produces two identical copies whose quality is independent of the input state. It was proved that the Bužek and Hillery UQCM is optimal by using the fidelity as measurement [4–7]. The general optimal quantum cloning transformation with N identical pure input states and M copies was proposed by Gisin and Massar [6,5]. And the no-cloning theorem was also extended to other cases [9,10].

The UQCM presented by Bužek, Hillery and Gisin, Massar is for the 2-level quantum system [3,6]. The 2-dimensional Hilbert space is spanned by 2 orthonormal basis vectors $|1\rangle, |2\rangle$ ($|\uparrow\rangle, |\downarrow\rangle$). For a d-level quantum system, the N to M optimal quantum cloning is formulated by Werner [7] and Keyl and Werner [8]. They have shown that the optimal cloning map to obtain M optimal clones from N identical input states is the projection of the direct product of the N input states and $M-N$ identity states onto the symmetric subspace of M particles. The optimal fidelity is obtained from the so-called Black Cow factor. Bužek and Hillery [11] presented the universal 1 to 2 quantum cloning transformation of states in d-dimensional Hilbert space. Albeverio and Fei [12] extended this result to 1 to M cloning, and a special case of N to M cloning in which the input state is a restricted N identical d-level quantum system. In this paper, generalizing the results in Ref. [11,12], we shall present the optimal N to M unitary cloning transformation for the d-level system. The fidelity achieves the optimal fidelity given by Werner [7]. Our results recover the previous results in Ref. [3,6,11,12] for special values of N, M and d. The cloning transformation presented in this paper should be the physical implementation of the optimal cloning map given in Ref. [7,8]. The result is useful in obtaining the quantum networks [13] and the remote information concentration [14].

A d-level quantum system is spanned by the orthonormal basis $|i\rangle$ with $i = 1, \cdots, d$. And an arbitrary pure state takes the form $|\Psi\rangle = \sum_{i=1}^{d} x_i|i\rangle$ with $\sum_{i=1}^{d}|x_i|^2 = 1$. The N to M quantum cloning means that we map by unitary transformation N pure input states

$$|\Psi\rangle^{\otimes N} \otimes R \equiv \sum_{\mathbf{n}=0}^{N} \sqrt{\frac{N!}{n_1! \cdots n_d!}} x_1^{n_1} \cdots n_d^{n_d} |\mathbf{n}\rangle \otimes R \tag{1}$$

to M copies in $|\mathbf{m}\rangle \otimes R_{\mathbf{nm}}$, where vector $\mathbf{n}$ denotes $n_1, \cdots, n_d$, $|\mathbf{n}\rangle = |n_1, \cdots n_d\rangle$ is a completely symmetric and normalized state with n_i systems in $|i\rangle$, this state is invariant under permutations of all N d-level qubits. R denotes $M-N$ blank copies and the initial state of the cloning machine, $R_{\mathbf{nm}}$ are internal states of the cloning machine, where $\sum_{\mathbf{n}=0}^{N}$ means sum over all variables under the condition $\sum_{i=1}^{d} n_i = N$, we also have $\sum_{i=1}^{d} m_i = M$. More explicit, we need to find unitary transformations U_{NM} to map $|\mathbf{n}\rangle$ to $|\mathbf{m}\rangle$. We use fidelity F to describe the quality of the copies $F = \langle\Psi|\rho^{out}|\Psi\rangle$, where ρ^{out} denotes reduced density operator of each output d-level qubit by taking partial trace over all but one output d-level qubits.

We propose the N to M quantum cloning transformation for d-level quantum system as follows,

$$U_{NM}|\mathbf{n}\rangle \otimes R = \sum_{\mathbf{j}=0}^{M-N} \alpha_{\mathbf{nj}}|\mathbf{n}+\mathbf{j}\rangle \otimes R_{\mathbf{j}}, \tag{2}$$

where $\mathbf{n}+\mathbf{j}=\mathbf{m}, i.e., \sum_{k=1}^{d} j_k = M-N$, $R_{\mathbf{j}}$ denotes the orthogonal normalized internal states of the UQCM, and

$$\alpha_{\mathbf{nj}} = \sqrt{\frac{(M-N)!(N+d-1)!}{(M+d-1)!}}\sqrt{\prod_{k=1}^{d}\frac{(n_k+j_k)!}{n_k!j_k!}}. \tag{3}$$

This explicit construction of the unitary cloning transformation is the essence of this paper. The dimension of the ancilla is $\frac{(M-N+d-1)!}{(M-N)!(d-1)!}$. The N to M cloning in Ref. [12] corresponds to the case $n_i = N$ in the above relations.

As an example, we present the N to M quantum cloning transformation for the 2-level system since it is simple and is the physically interesting case. Gisin and Massar's UQCM from N to M for the 2-level system is well known [6]. Our quantum cloning transformation turns out to be a different but an equivalent form. Suppose the pure input state takes the form $|\Psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$, the N identical input qubits are thus written as

$$|\Psi\rangle^{\otimes N} = \sum_{k=0}^{N} \alpha^{N-k}\beta^{k}\sqrt{C_N^k}|(N-k)\uparrow, k\downarrow\rangle, \tag{4}$$

where $|(N-k)\uparrow, k\downarrow\rangle$ denotes the symmetric and normalized state with $N-k$ qubits in the state $|\uparrow\rangle$ and k qubits in the state $|\downarrow\rangle$, and we have $C_N^k = \frac{N!}{(N-k)!k!}$ in standard notation. According to (2,3), we can write the quantum cloning transformation for the qubits case as,

$$\begin{aligned} U_{NM}|(N-k)\uparrow, k\downarrow\rangle \otimes R &= \sum_{j=0}^{M-N} \alpha_{kj}|(M-k-j)\uparrow, (k+j)\downarrow\rangle \otimes R_j, \\ \alpha_{kj} &= \sqrt{\frac{(M-N)!(N+1)!}{k!(N-k)!(M+1)!}}\sqrt{\frac{(k+j)!(M-k-j)!}{j!(M-N-j)!}}. \end{aligned} \tag{5}$$

The reduced density operator describing the state of each ouput is given by

$$\begin{aligned} \rho^{out} &= |\uparrow\rangle\langle\uparrow| \left(\sum_{j=0}^{M-N}\sum_{k=0}^{N} |\alpha|^{2N-2k}|\beta|^{2k}\alpha_{kj}^2 C_N^k \frac{M-k-j}{M} \right) \\ &+ |\uparrow\rangle\langle\downarrow| \left(\sum_{j=0}^{M-N}\sum_{k=0}^{N-1} |\alpha|^{2N-2k}|\beta|^{2k}\frac{\beta^*}{\alpha^*}\alpha_{kj}\alpha_{(k+1)j}\sqrt{C_N^k C_N^{k+1}}\frac{\sqrt{(k+j+1)(M-k-j)}}{M} \right) \\ &+ |\downarrow\rangle\langle\uparrow| \left(\sum_{j=0}^{M-N}\sum_{k=0}^{N-1} |\alpha|^{2N-2k}|\beta|^{2k}\frac{\beta}{\alpha}\alpha_{kj}\alpha_{(k+1)j}\sqrt{C_N^k C_N^{k+1}}\frac{\sqrt{(k+j+1)(M-k-j)}}{M} \right) \\ &+ |\downarrow\rangle\langle\downarrow| \left(\sum_{j=0}^{M-N}\sum_{k=0}^{N} |\alpha|^{2N-2k}|\beta|^{2k}\alpha_{kj}^2 C_N^k \frac{k+j}{M} \right). \end{aligned} \tag{6}$$

With the help of the definition, the fidelity of the quantum cloning transformation can thus be calculated as

$$F = \frac{MN+M+N}{M(N+2)}. \tag{7}$$

This is the optimal fidelity which can be achieved [6,5]. We thus see that the quantum cloning transformation (5) is optimal and is equivalent to the UQCM given by Gisin and Massar [6].

Finally, we present the formula for the d-level quantum system. The input states are N identical states given in (1). By use of the transformation in (2,3), the output reduced density operator at each d-level qubit can be written as

$$\rho^{out} = \sum_{i=1}^{d} |i\rangle\langle i| \left(\sum_{\mathbf{n}=0}^{N} \sum_{\mathbf{j}=0}^{M-N} (\prod_{k=1}^{d} \frac{|x_k|^{2n_k}(n_k+j_k)!}{(n_k!)^2 j_k!}) \frac{N!(M-N)!(N+d-1)!}{(M+d-1)!} \frac{(n_i+j_i)}{M} \right)$$
$$+ \sum_{i\neq l}^{d} |i\rangle\langle l| \left(\sum_{\mathbf{n}=0, n_l<N}^{N} \sum_{\mathbf{j}=0}^{M-N} (\prod_{k=1}^{d} \frac{|x_k|^{2n_k}(n_k+j_k)!}{(n_k!)^2 j_k!}) \frac{x_l^*}{x_i^*} \frac{N!(M-N)!(N+d-1)!}{(M+d-1)!M} \frac{n_i(n_l+j_l+1)}{n_l+1} \right). \tag{8}$$

The fidelity can be calculated as

$$F = \frac{N(d+M)+M-N}{(d+N)M}. \tag{9}$$

This is the optimal fidelity achieved by universal quantum cloning transformation for a d-level system given by Werner [7] and Keyl and Werner [8]. Thus the UQCM given in (2,3) is optimal. As the case of 1 to 2 UQCM [11], the cloning transformation in this letter can also be applied for the arbitrary impure states cloning by considering the input density operator as $\sum_{ij} A_{ij}|\Psi_i\rangle^{\otimes N}\ {}^{N\otimes}\langle\Psi_j|$. The fidelity keeps the same (9) as the case of pure input state.

To summarize, we have presented the optimal unitary cloning transformations with N identical unknown input states to M copies for the d-level system. As the results in Ref. [3,6–8,11,12], the output of the UQCM in this letter is completely symmetric. If we want to have a higher fidelity, for example, in the first output d-level qubit than the optimal fidelity, the fidelity in other output d-level qubits will decrease. We give a simple and limited example: the input are N identical states $|\Psi\rangle|\Psi\rangle^{\otimes N-1}$, in the cloning transformation, we keep the first d-level qubit unchanged while the other N-1 qubits are changed by the optimal cloning transformation. The fidelity for the first output d-level qubit is 1, while the fidelity of other d-level qubits equal to the case of $N-1$ to $M-1$ cloning whose fidelity is smaller than the case of N to M cloning. In the optimal UQCM (2), for a fixed M, because that the input $|\mathbf{n}\rangle$ always keeps in the output state $|\mathbf{m}\rangle$, we can see that the more quantum information is available, the better fidelity of the copies we can get. We remark that, similar to 2-dimensional case [6], the N to $N+1$ UQCM is much simple, the right hand side of (2) contains only d terms for every $|\mathbf{n}\rangle$, and the UQCM needs only d internal states altogether.

In order to construct the quantum networks to realize the optimal quantum cloning, the results in this paper which give the cloning transformations for all symmetric input states should be useful. Different from the quantum networks, the optimal quantum cloning realized via photon stimulated emission was proposed for the 2-level system in [15,16]. For the d-level system, a similar result should also be found with a generalized photon-atom interaction Hamiltonian.
Acknowlegements: One of the authors, H.F. acknowleges the support of JSPS, and the hospitality of Wadati group in Department of Physics, University of Tokyo where part of this work were done. We thank X.B.Wang and G.Weihs for useful discussions.

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